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FRACTIONAL FREE ELECTRON LASER EQUATION AND NEW GENERALIZATION OF GENERALIZED M-SERIES

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ABSTRACT

In this decade fractional free electron laser (FEL) equation are studied due to their utility and importance in mathematical physics, The aim of present paper is to find the solution of generalized fractional order free electron laser (FEL) equation, using New generalization of Generalized M-series .The results obtained here is moderately universal in nature. Special cases, relating to the exponential function is also considered.

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INTRODUCTION

The Fractional Free Electron Laser Equation:

The unsaturated behavior of the free electron laser (FEL) is governed by the following first order integro differential equation of Volterra – type [3,4]. :

$$D_T\, a(T) = -i\pi\, g_0 \int_0^T \xi\, a(T-\xi)\, e^{iv\xi}\, d\xi, \quad 0 \le T < 1 \qquad \ldots(1.1)$$

where T is a dimensionless time variable, g_0 is a positive constant known as the small–signal gain and the constant v is the detuning parameter. The functional (T) is a complex-field amplitude which is assumed to be dimensionless and satisfies the initial condition a (0) = 1. Here we employ the Riemann-Liouville definition of fractional integral equation defined by a simplified version of (1.1) changing the scale by putting $t = x\sigma$ and a = 0 this yields

$$R_x^{\alpha}\, f(x) \equiv I_x^{\alpha}\, f(x) = \frac{x^{\alpha}}{\Gamma(\alpha)} \int_0^1 (1-\sigma)^{\alpha-1}\, f(x\sigma)\, d\sigma, \quad \mathrm{Re}\,\sigma \ge 0 \qquad \ldots(1.2)$$

The definition (1.2) can be written as

$$R_x^{\alpha}\, f(x) \equiv I_x^{\alpha}\, f(x) = \frac{d^n}{dx^n} R_x^{\alpha+n}\, f(x), \qquad \mathrm{Re}\,(\alpha + n) > 0 \qquad \ldots(1.3)$$

Boyadjiev et al. [3] have treated a non homogeneous case of (1.2) in which the ordinary first derivative D_T is replaced by the fractional D_T^{α} with $\alpha > 0$, that is

$$D_T^{\alpha}\, a(T) = \lambda \int_0^T t\, a(T-t)\, e^{ivt}\, dt + \beta\, e^{ivt}, \quad 0 \le T \le 1 \qquad \ldots(1.4)$$

with $\beta, \lambda, \in C$ and $v \in R$. Furthermore the following generalization of (1.4) has been considered by the authors [2]

$$D_T^{\alpha}\, a(T) = \lambda \int_0^T t^{\delta}\, a(T-t)\, e^{ivt}\, dt + \beta\, e^{ivt}, \quad 0 \le T \le 1 \qquad \ldots(1.5)$$

where $\beta, \lambda, \in \mathrm{C}, \mathrm{v} \in \mathrm{R}$ and $\delta > -1$, In the present section, we investigate a further generalization of equation (1.5), whereby the exponential term is replaced by the M-SERIES

The New Generalization of Generalized M-Series

Here , first the notation and the definition of the New Generalization of Generalized M-series, introduced by Ahmad Faraj , Tariq Salim , Safaa Sadek, Jamal Ismai [6] has been given as

$$\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(a_1...,a_p;b_1,...,b_q;z) = \underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z),$$

$$\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z) = \sum_{k=0}^{\infty} \frac{(a_1)_{km}...(a_p)_{km}}{(b_1)_{kn}...(b_q)_{kn}} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad ....(2.1)$$

Here $\alpha, \beta \in C, \mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\beta) > 0$, $(a_j)_{km}$, $(b_j)_{kn}$ are the pochammer symbols and m,n are non-negative real numbers.

The Generalized Equation

The generalization of equation (1.5) obtained by replacing e^{ivt} by

$\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z) = \sum_{k=0}^{\infty} \frac{(a_1)_{km}...(a_p)_{km}}{(b_1)_{kn}...(b_q)_{kn}} \frac{z^k}{\Gamma(\alpha k + \beta)}$ takes the form

$$D_T^{\alpha}\, a(T) = \lambda \int_0^T t^{\delta}\, a(T-t) \underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(ivt)\, dt + \beta \underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(ivt), \quad ...(3.1)$$

$$0 \le T \le 1$$

where $\alpha, \beta, \lambda \in \mathrm{C}; \mathrm{v} \in \mathrm{R},\ \alpha > 0,\ \delta > -1,$ Given real numbers b_k, the corresponding initial conditions are :

$$D_T^{\alpha-k}\, a(T)\Big|_{T=0} = b_k,\ \ k = 1, 2, 3, ..., N \quad ...(3.2)$$

with $N = [\alpha] + 1$, so that $N - 1 \le \alpha < N$. Equation (1.5) can be deduced from (3.1) by taking $\alpha = 1$ while (3.4) follows when $\alpha = 1$, $\delta = 1$ To obtain the solution of (3.1) for the given initial conditions (3.2), we use (3.2).

Let $\xi = T - t$ in (3.1), so that $D_T^{\alpha}\, a(T) = \lambda \int_0^T (T-\xi)^{\delta} a(\xi) \underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(iv(t-\xi)d\xi + \beta \underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(ivT)$ ...(3.3)

Using the series representation for $\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z)$ we get

$$a(T) = \mathrm{a}_0\,(\mathrm{T}) + \lambda\, \mathrm{I}_{\mathrm{T}}^{\alpha} \sum_{\mathrm{k}=0}^{\infty} \frac{(\mathrm{a}_1)_{km} \ldots (a_p)_{km}}{(\mathrm{b}_1)_{kn} \ldots (b_q)_{kn}} \frac{(\mathrm{iv})^{\mathrm{k}}}{\Gamma(\alpha\mathrm{k}+\beta)} \int_0^{\mathrm{T}} (\mathrm{T}-\xi)^{\delta+k}\, \mathrm{a}\,(\xi)\, \mathrm{d}\xi \Bigg]$$

$$+ \beta\, \mathrm{I}_{\mathrm{T}}^{\alpha} \left[\sum_{\mathrm{k}=0}^{\infty} \frac{(\mathrm{a}_1)_{km} \ldots (a_p)_{km}}{(\mathrm{b}_1)_{kn} \ldots (b_q)_{kn}} \frac{(\mathrm{iv})^{\mathrm{k}}}{\Gamma(\alpha\mathrm{k}+\beta)} \right] \quad \ldots(3.4)$$

where $$\mathrm{a}_0\,(\mathrm{T}) = \sum_{k=1}^{N} \frac{b_k}{\Gamma(\alpha - k + 1)} \mathrm{T}^{\mathrm{N-k}} \quad \ldots(3.5)$$

$$\int_0^t \int_0^T u\,(\mathrm{T},\mathrm{s})\,\mathrm{ds}\,\mathrm{dT} = \int_0^t \int_s^t \mathrm{u}\,(\mathrm{T},\mathrm{s})\,\mathrm{dT}\,\mathrm{ds} \quad \ldots(3.6)$$

We obtain the following result

$$a\,(\mathrm{T}) = \mathrm{a}_0\,(\mathrm{T}) + \frac{\lambda}{\Gamma(\alpha+1)} \int_0^T \mathrm{a}\,(\xi)\,(\mathrm{T}-\xi)^{\alpha+\delta} \overset{\alpha,\beta}{\underset{p,q;m,n}{M}} (\mathrm{i\,v}\,(\mathrm{T}-\xi))\mathrm{d}\xi$$

$$+ \frac{\beta\,\Gamma(\mathrm{k}+1)\,\mathrm{T}^{\alpha}}{\Gamma(\alpha+k+1)} \overset{\alpha,\beta}{\underset{p,q;m,n}{M}} (\mathrm{ivT}) \quad \ldots(3.7)$$

Since (37) is a Volterra integral equation with continuous kernel, it admits a unique continuous solution (2.4). Finally, we consider some special cases of the generalized fractional integro-differential equation of volterra – type (3.1)

If $\alpha = 1, \beta = 1$ and there is no upper and lower parameter

$$D_T^{\alpha}\ \mathrm{a}\,(\mathrm{T}) = \lambda \int_0^{\mathrm{T}} t^{\delta}\ \mathrm{a}\,(\mathrm{T}-\mathrm{t})\,\mathrm{e}^{\mathrm{iv}\,(\mathrm{T}-\xi)\,\mathrm{d}\xi} + \frac{\beta\,\Gamma(\mathrm{k}+1)\,\mathrm{T}^{\alpha}\ \mathrm{e}^{\mathrm{ivt}}}{\Gamma(\alpha+\mathrm{k}+1)} \quad \ldots(3.8)$$

Equivalently

$$D_T^{\alpha}\ \mathrm{a(T)} = \lambda \int_0^{\mathrm{T}} t^{\delta}\ \mathrm{a}\,(\mathrm{T}-\mathrm{t})\ {}_0\mathrm{F}_0\,(\text{-;-;}\,\mathrm{iv}\,(\mathrm{T}-\xi)) +$$

$$\beta \frac{\Gamma(\mathrm{k}+1)}{\Gamma(\alpha+\mathrm{K}+1)} \mathrm{T}^{\alpha}\ {}_0\mathrm{F}_0\,(\text{-;-;ivT}) \quad \ldots(3.9)$$

CONCLUSION

In this present work, we have introduced a fractional generalization of the standard free electron laser (FEL) equation The results of the Advanced generalized fractional free electron laser (FEL) equation and its special cease are same as the results of Al-Shammery, A, Kalla, and Khajah, [2](2003).

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